

Hamilton-Jacobi Equation and the Tree Formula for Proper Vertices

Chr. Wiecekowsky

Inst. f. Theor. Physik I, Universität Münster,
Wilhelm-Klemm-Str. 9, 48159 Münster, Germany

September 24, 1998

1 Introduction

Proper vertices are central elements of a Statistical Field Theory. See [1] and references therein. They are related to the connected vertices (or correlators) through a tree formula. This fact is both well known and ubiquitously used in the literature. In pedagogical accounts, the tree formula is usually motivated by a low order computation (in powers of fields) but neither written down explicitly nor proved to all orders. In this note, we intend to fill this pedagogical gap. Our proof is an elaboration of a remark in [2]. The proof is complete aside of convexity and stability issues needed to have well defined Legendre transforms and path integrals. Trivial normalization constants will be neglected.

The generating functions of the proper vertices and the connected vertices are Legendre transforms of each other:

$$\Gamma(\Phi) = \sup_J \left\{ (\Phi, J) - W(J) \right\} \quad (1)$$

For definiteness, we consider the case of a real scalar field theory on a finite lattice $\Lambda \subset \mathbb{Z}^D$, where $\Phi, J \in \{\phi : \Lambda \rightarrow \mathbb{R}\}$ and $(\Phi, J) = \sum_{x \in \Lambda} \Phi(x)J(x)$. In Statistical Field Theory, the generating function $W(J)$ comes as a path integral

$$W(J) = \hbar \log \left[\int \mathcal{D}\phi \exp \left[\frac{1}{\hbar} \left\{ (\phi, J) - S(\phi) \right\} \right] \right] \quad (2)$$

in terms of the classical action $S(\phi)$. In the classical limit, when $\hbar \rightarrow 0$, the path integral becomes a Legendre transform. And, since the Legendre transform is involutive, (1) inverts the classical limit of (2). The classical limit of $\Gamma(\Phi)$ is thus $S(\phi)$. The tree formula follows from an analogous path integral representation for the inverse of (1),

$$W(J) = \lim_{\hbar \rightarrow 0} \hbar \log \left[\int \mathcal{D}\Phi \exp \left[\frac{1}{\hbar} \left\{ (\Phi, J) - \Gamma(\Phi) \right\} \right] \right]. \quad (3)$$

Of course, \hbar is here only an auxiliary variable cousin of the constant of nature. One then writes $\Gamma(\Phi) = \frac{1}{2}(\Phi, G^{-1}\Phi) + \Gamma_1(\Phi)$ and performs a perturbation expansion. The propagator is G , the interaction is $\Gamma_1(\Phi)$. Only tree diagrams contribute. To lowest orders (in powers of $\Gamma_1(\Phi)$), this perturbation expansion reads

$$\begin{aligned} W(J) = & \frac{1}{2}(J, GJ) - \Gamma_1(GJ) + \frac{1}{2} \sum_{x_1, y_1 \in \Lambda} G(x_1, y_1) \Gamma_{1, x_1}(GJ) \Gamma_{1, y_1}(GJ) \\ & - \frac{1}{3!} \sum_{x_1, y_1, x_2, y_2 \in \Lambda} G(x_1, y_1) G(x_2, y_2) \left\{ \Gamma_{1, x_1}(GJ) \Gamma_{1, y_1 x_2}(GJ) \Gamma_{1, y_2}(GJ) \right. \\ & \left. + \Gamma_{1, x_1 x_2}(GJ) \Gamma_{1, y_1}(GJ) \Gamma_{1, y_2}(GJ) + \Gamma_{1, x_1}(GJ) \Gamma_{1, x_2}(GJ) \Gamma_{1, y_1 y_2}(GJ) \right\} \\ & + O(\Gamma_1^4) \end{aligned} \quad (4)$$

where

$$\Gamma_{1, x_1 \dots x_n}(\Phi) = \frac{\partial^n}{\partial \Phi(x_1) \dots \partial \Phi(x_n)} \Gamma_1(\Phi). \quad (5)$$

We will show that this series, beginning as in (4), is the iterative solution to a Hamilton-Jacobi equation. We will then write down explicitly the terms of any order. With this expression in our hands, we will demonstrate that the tree expansion converges.

2 Hamilton-Jacobi Equation

Our first step is to write (3) in a fancy way. From the fancy formula, we deduce a Hamilton-Jacobi differential equation. More details (and references) thereabout can be found in [3].

2.1 Propagator G

The tree formula involves lines. These lines are propagators G . They correspond to quadratic (or kinetic) terms in $\Gamma(\Phi)$ and $W(J)$, which are brotherly split from $\Gamma(\Phi) = \Gamma_0(\Phi) + \Gamma_1(\Phi)$ and $W(J) = W_0(J) + W_1(J)$:

$$\Gamma_0(\Phi) = \frac{1}{2}(\Phi, G^{-1}\Phi), \quad W_0(J) = \frac{1}{2}(J, GJ). \quad (6)$$

Usually, but not necessarily, $W_0(J)$ is the full quadratic term of $W(J)$, and $G(x, y)$ is the (full) connected two point function of the theory:

$$G(x, y) = \frac{\partial^2}{\partial J(x) \partial J(y)} W(J) \Big|_{J=0} \quad (7)$$

However, there is nothing wrong with leaving behind a quadratic remainder in $W_1(J)$, which then adds a quadratic interaction. To be on safe ground, we will assume that $G(x, y)$ is real, symmetric, and (that $W_0(J)$ is) positive. Then G is invertible, and this inverse appears in $\Gamma_0(\Phi)$. By completing a square, (3) becomes

$$W_1(J) = \lim_{\hbar \rightarrow 0} \hbar \log \left[\int d\mu_{\hbar G}(\Phi) \exp \left\{ -\frac{1}{\hbar} \Gamma_1(\Phi + GJ) \right\} \right] \quad (8)$$

where

$$d\mu_G(\Phi) = \det(2\pi G)^{-\frac{1}{2}} \mathcal{D}\Phi \exp \left\{ -\frac{1}{2}(\Phi, G^{-1}\Phi) \right\} \quad (9)$$

is the Gaussian measure on field space associated with the propagator G . In (8), the external source J is dressed with the propagator G . It is convenient to introduce a new symbol for this dressed source: $\Psi = GJ$. Formula (8) is the optimal starting point for the perturbation expansion (4).

2.2 Interpolation parameter

Our second step is to tune up \hbar temporarily to finite values in (8). Moreover, we join to \hbar a second interpolation parameter t , which takes values in between zero and one:

$$W_1(\Psi, t) = \hbar \log \left[\int d\mu_{t\hbar G}(\Phi) \exp \left\{ -\frac{1}{\hbar} \Gamma_1(\Phi + \Psi) \right\} \right] \quad (10)$$

A look at (8) reveals that \hbar appears at two places in the integral. The idea with the new interpolation parameter is to tune both \hbar s independently. The interpolated quantity solves the differential equation

$$\frac{\partial}{\partial t} W_1(\Psi, t) = \frac{1}{2} \left(\frac{\partial}{\partial \Psi} W_1(\Psi, t), G \frac{\partial}{\partial \Psi} W_1(\Psi, t) \right) - \frac{\hbar}{2} \left(\frac{\partial}{\partial \Psi}, G \frac{\partial}{\partial \Psi} \right) W_1(\Psi, t) \quad (11)$$

with the initial condition $W_1(\Psi, 0) = -\Gamma_1(\Psi)$. The aim of the game is $W_1(\Psi, 1)$ at $\hbar = 0$. But (11) is a perfect instant to put $\hbar = 0$, remaining with only one interpolation parameter t . The result is the Hamilton-Jacobi

$$\frac{\partial}{\partial t} W_1(\Psi, t) = \frac{1}{2} \left(\frac{\partial}{\partial \Psi} W_1(\Psi, t), G \frac{\partial}{\partial \Psi} W_1(\Psi, t) \right). \quad (12)$$

The initial condition is independent of \hbar . Therefore (12) is integrated to

$$W_1(\Psi, t) = \frac{1}{2} \int_0^t ds \left(\frac{\partial}{\partial \Psi} W_1(\Psi, s), G \frac{\partial}{\partial \Psi} W_1(\Psi, s) \right) - \Gamma_1(\Psi) \quad (13)$$

Remarkably, the solution (13) to the Hamilton-Jacobi equation (12) performs a Legendre transformation of the initial data. The Legendre transform is recovered as the boundary value

$$W_1(J) = W_1(GJ, 1) \quad (14)$$

The iteration of (13) generates a tree formula. Indeed, the bilinear term creates a branch in every iteration step.

3 Tree expansion

A basic version of the tree expansion is obtained as follows. We develop $W_1(\Psi, t)$ into a power series in t ,

$$W_1(\Psi, t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} W_1^{(n)}(\Psi). \quad (15)$$

The order zero is our initial condition $W_1^{(0)}(\Psi) = -\Gamma_1(\Psi)$. The higher orders are recursively determined by

$$W_1^{(n+1)}(\Psi) = \frac{1}{2} \sum_{m=0}^n \binom{n}{m} \left(\frac{\partial}{\partial \Psi} W_1^{(m)}(\Psi), G \frac{\partial}{\partial \Psi} W_1^{(n-m)}(\Psi) \right). \quad (16)$$

The illustrative formula (4) is found by iterating (16) a few times. The outcome of an infinite iteration of (16) is the renowned tree formula

$$W_1^{(n)}(\Psi) = \frac{(-1)^{n+1}}{n+1} \sum_{\tau \in T_n(\{1,2,\dots,n+1\})} \Delta_\Psi(\tau) \prod_{m=1}^{n+1} \Gamma(\Psi_m) \Big|_{\Psi_1=\dots=\Psi_{n+1}=\Psi} \quad (17)$$

Here $T_n(\{1,2,\dots,n+1\})$ denotes the set of n -trees on the set $\{1,2,\dots,n+1\}$. An n -tree is a set of links

$$\tau = \{\{i_1, j_1\}, \dots, \{i_n, j_n\}\}, \quad (18)$$

consisting of n sets $\{i, j\}$ (the links) with $i, j \in \{1,2,\dots,n+1\}$ and $i \neq j$ such that

$$\{1,2,\dots,n+1\} = \bigcup_{a=1}^n \{i_a, j_a\} \quad (19)$$

A less formal explanation is this: draw $n+1$ dots on a sheet of paper; connect the dots pairwise with n links such that the resulting diagram is connected; then this is a tree diagram. If you happen to draw a loop, the resulting diagram will be disconnected. With each n -tree is associated a linking operator

$$\Delta_\Psi(\tau) = \prod_{a=1}^n \left(\frac{\partial}{\partial \Psi_{i_a}}, G \frac{\partial}{\partial \Psi_{j_a}} \right). \quad (20)$$

As its name reveals, the linking operator links together the product of proper vertices in (17).

3.1 Examples

There are three 2-trees on $\{1,2,3\}$:

$$\{\{1,2\}, \{2,3\}\}, \quad \{\{1,3\}, \{2,3\}\}, \quad \{\{1,2\}, \{1,3\}\}.$$

There are sixteen 3-trees on $\{1,2,3,4\}$: twelve straight trees

$$\begin{aligned} &\{\{1,2\}, \{2,3\}, \{3,4\}\}, \quad \{\{1,2\}, \{2,3\}, \{1,4\}\}, \\ &\{\{1,2\}, \{1,3\}, \{2,4\}\}, \quad \{\{1,2\}, \{1,4\}, \{3,4\}\}, \\ &\{\{1,3\}, \{1,4\}, \{2,3\}\}, \quad \{\{1,3\}, \{2,3\}, \{2,4\}\}, \\ &\{\{1,2\}, \{1,3\}, \{3,4\}\}, \quad \{\{1,4\}, \{2,3\}, \{3,4\}\}, \\ &\{\{1,3\}, \{2,4\}, \{3,4\}\}, \quad \{\{1,2\}, \{1,4\}, \{3,4\}\}, \\ &\{\{1,4\}, \{2,3\}, \{2,4\}\}, \quad \{\{1,2\}, \{1,4\}, \{2,4\}\}, \end{aligned}$$

and four trees with a fork

$$\begin{aligned} &\{\{1,2\}, \{1,3\}, \{1,4\}\}, \quad \{\{2,1\}, \{2,3\}, \{2,4\}\}, \\ &\{\{3,1\}, \{3,2\}, \{3,4\}\}, \quad \{\{4,1\}, \{4,2\}, \{4,3\}\}. \end{aligned}$$

3.2 Proof by Induction on n

The trivial tree is only a single dot. It suffices therefore to prove the induction step $n \rightarrow n+1$. From the induction hypothesis, we find

$$\begin{aligned}
W_1^{(n+1)}(\Psi) &= \frac{(-1)^{n+2}}{2(n+1)(n+2)} \sum_{m=0}^n \binom{n+2}{m+1} \\
&\sum_{i=1}^{m+1} \sum_{j=m+2}^{n+2} \sum_{\tau_1 \in T_m(\{1,2,\dots,m+1\})} \sum_{\tau_2 \in T_{n-m}(\{m+2,m+3,\dots,n+2\})} \\
&\left(\frac{\partial}{\partial \Psi_i}, G \frac{\partial}{\partial \Psi_j} \right) \Delta_{\Psi}(\tau_1) \Delta_{\Psi}(\tau_2) \prod_{i=1}^{n+2} \Gamma(\Psi_i) \Big|_{\Psi_1=\dots=\Psi_{n+2}=\Psi}
\end{aligned} \tag{21}$$

But $\binom{n+2}{m+1}$ is just the number of subsets of $\{1, 2, \dots, n+2\}$ with $m+1$ elements. Therefore,

$$\begin{aligned}
W_1^{(n+1)}(\Psi) &= \frac{(-1)^{n+2}}{2(n+1)(n+2)} \sum_{m=0}^n \sum_{\{1,2,\dots,n+2\}=I \cup J} \delta_{|I|,m+1} \delta_{|J|,n-m+1} \\
&\sum_{i \in I} \sum_{j \in J} \sum_{\tau_1 \in T_m(I)} \sum_{\tau_2 \in T_{n-m}(J)} \left(\frac{\partial}{\partial \Psi_i}, G \frac{\partial}{\partial \Psi_j} \right) \Delta_{\Psi}(\tau_1) \Delta_{\Psi}(\tau_2) \\
&\prod_{i=1}^{n+2} \Gamma(\Psi_i) \Big|_{\Psi_1=\dots=\Psi_{n+2}=\Psi}
\end{aligned} \tag{22}$$

The factor $\frac{1}{2}$ takes care of the fact that (I, J) and (J, I) yield equal contributions. The factor $\frac{1}{n+1}$ takes care of the fact that any $(n+1)$ -tree on $\{1, 2, \dots, n+2\}$ has (tautologically) $n+1$ links, and can therefore be broken up into $n+1$ pairs of subtrees by cutting a line. Eq. (22) conversely assembles $(n+1)$ -trees in all of these possible ways. The result is

$$W_1^{(n+1)}(\Psi) = \frac{(-1)^{n+2}}{n+2} \sum_{\tau \in T_{n+1}(\{1,2,\dots,n+2\})} \Delta_{\Psi}(\tau) \prod_{i=1}^{n+2} \Gamma(\Psi_i) \Big|_{\Psi_1=\dots=\Psi_{n+2}=\Psi} \tag{23}$$

The induction step is complete. \square

The result of the tree expansion is the following tree formula for the Legendre transform:

$$W(J) = \frac{1}{2}(J, GJ) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \sum_{\tau \in T_{n-1}(1,2,\dots,n)} \Delta_{\Psi}(\tau) \prod_{i=1}^n \Gamma_1(\Psi_i) \Big|_{\Psi_1=\dots=\Psi_n=GJ} \tag{24}$$

Quite remarkably, the different looking formulas (1), (13), and (24) are variants of the same Legendre transform.

4 Convergence of the Tree Expansion

There remains the question whether (24) is formal expression. In perturbation theory, this question is irrelevant since always only finitely many terms in the tree expansion contribute to finite order expressions in perturbation theory. But the perturbation (or loop) expansion is another expansion on top of the tree expansion.

The Hamilton-Jacobi equation (12) is an ideal tool to study the convergence of the tree expansion. The first thing to do is to choose a suitable norm. A very suitable norm is

$$\|W_1(\cdot, t)\|_h = \sum_{n=0}^{\infty} \frac{h^n}{n!} \sup_{x_1 \in \Lambda} \sum_{x_2, \dots, x_n \in \Lambda} |W_{1, x_1 \dots x_n}(0, t)|. \quad (25)$$

The meaning of h is a radius of analyticity in source space. The point with this norm is that (12) immediately implies a differential inequality, namely

$$\frac{\partial}{\partial t} \|W_1(\cdot, t)\|_h \leq \frac{\|G\|}{2} \left(\frac{\partial}{\partial h} \|W_1(\cdot, t)\|_h \right)^2, \quad (26)$$

where

$$\|G\| = \sup_{x \in \Lambda} \sum_{y \in \Lambda} |G(x, y)|. \quad (27)$$

We will assume that $\|G\|$ is finite. (On a finite lattice, this condition is per se fulfilled.) Any solution of (26) is majorized by the solution to the simpler Hamilton-Jacobi equation

$$\frac{\partial}{\partial t} f(t, h) = \frac{\|G\|}{2} \left(\frac{\partial}{\partial h} f(t, h) \right)^2. \quad (28)$$

The initial condition $W_1(\Psi, 0) = -\Gamma_1(\Psi)$ gives us the initial condition $f(0, h) = \|\Gamma_1\|_h$ for (28). Depending on the properties of this initial value, one ascertains the existence (and an estimate on) the boundary value $f(1, h)$. Let us only quote the following result from [4]:

If $\|\Gamma_1\|_h$ is holomorphic on a disc $\{h \in \mathbb{C} : |h| \leq h_0\}$, and if $\|\Gamma_1\|_{h_0} \leq \frac{(h_0 - h_1)^2}{16\|G\|}$ for some $h_1 \in (0, h_0)$, then $\|W_1(\cdot, 1)\|_h$ is holomorphic on the (smaller) disc $\{h \in \mathbb{C} : |h| \leq h_1\}$, and it satisfies the bound $\|W_1(\cdot, 1)\|_{h_1} \leq \|\Gamma_1\|_{h_0}$.

5 Concluding Remarks

We have shown that the Legendre transform can be advantageously studied as the solution of a Hamilton-Jacobi equation. Both the tree formula and an estimate on the tree expansion are straight forward consequences of the Hamilton-Jacobi equation.

Needless to say that this method also applies to field theories with a more complicated field content. It is immediately applicable to the case of Fermions. With some extra effort, all formulas can be extended to the continuum limit.

Since the Legendre transform is involutive, one immediately also has a tree formula for $\Gamma(\Phi)$ in terms of $W(J)$. All that has to be changed is that G has to be replaced by its inverse G^{-1} .

Although there are good reasons to study the proper vertices in Statistical Field Theory, modifications of the Hamilton-Jacobi formulation of the Legendre transform might prove to be useful in other contexts.

Graphical representation

The Hamilton-Jacobi equation (12) has the following graphical representation (expanded in powers of fields):

The diagram shows the graphical representation of the Hamilton-Jacobi equation. On the left, the derivative $\frac{\partial}{\partial t}$ is applied to a circle with n external lines (labeled 1, ..., n). This is equal to $\frac{1}{2} \sum_{m=0}^n \binom{n}{m}$ times a circle with m external lines (labeled 1, ..., m), minus a circle with m external lines (labeled 1, ..., m) connected to a shaded circle, which is then connected to a circle with $n-m$ external lines (labeled $m+1$, ..., n).

Acknowledgements

Thanks to Jörg Lemm, Gernot Münster, Martin Rehwald, and Manfred Salmhofer for pedagogical advice.

References

- [1] J. Zinn-Justin, Quantum Field Theory and Critical Phenomena, Oxford Science Publications 1990
- [2] S. Coleman, Laws of Hadronic Matter, Erice 1973
- [3] D. Brydges, Functional Integrals and their Applications, Lausanne 1992
- [4] D. Brydges, H. Yau, Grad ϕ Perturbations of Massless Gaussian Fields, Commun. Math. Phys. 129, 351-392 (1990)